THE AUTOMATIC ADDITIVITY OF $\xi-$LIE DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. It is shown that every nonlinear $\xi-$Lie derivation ($\xi \neq 1$) on $\mathcal{M}$ is an additive derivation.

1. Introduction and main results

Let $\mathcal{A}$ be an associate ring (or an algebra over a field $\mathbb{F}$). Then $\mathcal{A}$ is a Lie ring (Lie algebra) under the product $[x,y] = xy - yx$, i.e., the commutator of x and y. Recall that an additive (linear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called an additive (linear) derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. Derivations are very important maps both in theory and in applications, and have been studied intensively (see [8, 20, 21, 22] and the references therein). More generally, an additive (linear) map L from $\mathcal{A}$ into itself is called an additive (linear) Lie derivation if $L([x,y]) = [L(x), y] + [x, L(y)]$ for all $x, y \in \mathcal{A}$. The questions of characterizing Lie derivations and revealing the relationship between Lie derivations and derivations have received many mathematicians' attention recently (see [4, 9, 12, 16]). Very roughly speaking, additive (linear) Lie derivations in the context prime rings (operator algebras) can be decomposed as $\sigma + \tau$, where σ is an additive (linear) derivation and τ is an additive (linear) map sending commutators into zero. Similarly, associated with the Jordan product $xy + yx$. we have the conception of Jordan derivation which is also studied intensively (see [5, 6, 9] and the references therein).

Note that an important relation associated with the Lie product is the commutativity. Two elements x, y in an algebra $\mathcal{A}$ are commutative if $xy = yx$, that is, their Lie product is zero. More generally, if ξ is a scalar and if $xy = \xi yx$, we say that x commutes with y up to a factor ξ. The notion of commutativity up to a factor for pairs of operators is also important and has been studied in the context of operator algebras and quantum groups (Refs. [7, 11]). Motivated by this, the authors introduce a binary operation $[x,y]_\xi = xy - \xi yx$, called ξ-Lie product of x, y (Ref. [17]). This product is found playing a more and more important role in some research topics, and its study has recently attracted many authors attention (for example, see [17, 18]). Then it is natural to introduce the concept of ξ-Lie

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derivation. An additive (linear) map L from $\mathcal{A}$ into itself is called a $\xi-$Lie derivation if $L([x,y]_\xi) = [L(x),y]_\xi + [x,L(y)]_\xi$ for all $x,y \in \mathcal{A}$. This concept unifies several well-known notions. It is clear that a $\xi-$Lie derivation is a derivation if $\xi = 0$; is a Lie derivation if $\xi = 1$; is a Jordan derivation if $\xi = -1$. In [18], Qi and Hou characterized the additive $\xi-$Lie derivation on nest algebras.

Let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map (without the additivity or linearity assumption). We say that Φ is a nonlinear $\xi-$Lie derivation if $\Phi([x,y]_\xi) = [\Phi(x),y]_\xi + [x,\Phi(y)]_\xi$ for all $x,y \in \mathcal{A}$. Recently, Yu and Zhang [24] described nonlinear Lie derivation on triangular algebras. The aim of this note is to investigate nonlinear $\xi-$Lie derivations on von Neumann algebras ($\xi \neq 1$) and to reveal the relationship between such nonlinear $\xi-$Lie derivations and additive derivations. Due to vital importance of derivations, we firstly investigate nonlinear derivations. To our surprising, nonlinear derivations are automatically additive. Our main results read as follows.

Theorem 1.1. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. If $\Phi : \mathcal{M} \to \mathcal{M}$ is a nonlinear derivation, then Φ is an additive derivation.*

The following result reveals the relationship between general nonlinear $\xi-$Lie derivations and additive derivations.

Theorem 1.2. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. If ξ is a scalar not equal $0,1$ and $\Phi : \mathcal{M} \to \mathcal{M}$ is a nonlinear $\xi-$Lie derivation, then Φ is an additive derivation and $\Phi(\xi T) = \xi\Phi(T)$ for all $T \in \mathcal{M}$.*

It is worth mentioning that, as it turns out from Theorems 1.1 and Theorem 1.2, the additive structure and $\xi-$Lie multiplicative structure of von Neumann algebra with no central summands of type I_1 are very closely related to each other. We remark that the question when a multiplicative map is necessary additive is important in quantum mechanics and mathematics, and was discussed for associative rings in the purely algebraic setting ([14], for a recent systematic account, see [2]). In recent years, there is a growing interest in studying the automatic additivity of maps determined by the action on the product (see [1, 2, 13, 19, 23] and the references therein). We also remark that if $\xi = 1$, then $\xi-$Lie derivation is in fact a Lie derivation, while Lie derivation is not necessary additive. For example, let σ is an additive derivation of $\mathcal{M}$ and τ is a mapping of $\mathcal{M}$ into its center $\mathcal{Z}_\mathcal{M}$ which maps commutators into zero. Then $\sigma + \tau$ is a Lie derivation and such Lie derivation is not additive in general.

2. Notations and Preliminaries

Before embarking on the proof of our main results, we need some notations and preliminaries about von Neumann algebras. A von Neumann algebra $\mathcal{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. The set $\mathcal{Z}_\mathcal{M} = \{S \in \mathcal{M} \mid ST = TS \text{ for all } T \in \mathcal{M}\}$ is called the center of $\mathcal{M}$. For $A \in \mathcal{M}$, the central carrier of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = A$. It is well known that the central carrier of A is the projection with the range $[\mathcal{M}A(H)]$, the closed linear span of $\{MA(x) \mid M \in \mathcal{M}, x \in H\}$. For each self-adjoint operator $A \in \mathcal{M}$, we define the central core of A, denoted by $\underline{A}$, to be $\sup\{S \in \mathcal{Z}_\mathcal{M} \mid S = S^*, S \leq A\}$. Clearly, one has

$A-\underline{A}\geq 0$. Further if $S\in\mathcal{Z}_{\mathcal{M}}$ and $A-\underline{A}\geq S\geq 0$ then $S=0$. If P is a projection it is clear that $\underline{P}$ is the largest central projection $\leq P$. We call a projection core-free if $\underline{P}=0$. It is easy to see that $\underline{P}=0$ if and only if $\overline{I-P}=I$, here $\overline{I-P}$ denotes the central carrier of $I-P$. We use [10] as a general reference for the theory of von Neumann algebras.

In the following, there are several fundamental properties of von Neumann algebras from [3, 15] which will be used frequently. For convenience, we list them in a lemma.

Lemma 2.1. Let $\mathcal{M}$ be a von Neumann algebra.

(i) ([15, Lemma 4]) If $\mathcal{M}$ has no summands of type I_1, then each nonzero central projection of $\mathcal{M}$ is the central carrier of a core-free projection of $\mathcal{M}$;

(ii) ([3, Lemma 2.6]) If $\mathcal{M}$ has no summands of type I_1, then $\mathcal{M}$ equals the ideal of $\mathcal{M}$ generated by all commutators in $\mathcal{M}$.

By Lemma 2.1(i), one can find a non-trivial core-free projection with central carrier I, denoted by P_1. Throughout this paper, P_1 is fixed. Write $P_2=I-P_1$. By the definition of central core and central carrier, P_2 is also core-free and $\overline{P_2}=I$. According to the two-side Pierce decomposition of $\mathcal{M}$ relative P_1, denote $\mathcal{M}_{ij}=P_i\mathcal{M}P_j$, $i,j=1,2$, then we may write $\mathcal{M}=\mathcal{M}_{11}+\mathcal{M}_{12}+\mathcal{M}_{21}+\mathcal{M}_{22}$. In all that follows, when we write T_{ij}, S_{ij}, M_{ij}, it indicates that they are contained in $\mathcal{M}_{ij}$. A conclusion which is used frequently is $TM_{ij}=0$ for every $M_{ij}\in\mathcal{M}_{ij}$ implies that $TP_i=0$. Indeed $TP_iMP_j=0$ for all $M\in\mathcal{M}$ together with $\overline{P_j}=I$ gives $TP_i=0$. Similarly, if $M_{ij}T=0$ for every $M_{ij}\in\mathcal{M}_{ij}$, then $T^*M_{ij}^*=0$ and so $P_jT=0$. If $Z\in\mathcal{Z}_{\mathcal{M}}$ and $ZP_i=0$, then $ZMP_i=0$ for all $M\in\mathcal{M}$ which implies $Z=0$.

The next lemma is technical which plays an important role in the proof of Theorem 1.2.

Lemma 2.2. Let $T\in\mathcal{M}$, $\xi\neq 0,1$. Then $T\in\mathcal{M}_{ij}+(\xi P_i+P_j)\mathcal{Z}_{\mathcal{M}}$ $(1\leq i\neq j\leq 2)$ if and only if $[T,M_{ij}]_\xi=0$ for every $M_{ij}\in\mathcal{M}_{ij}$;

Proof. The necessity is clear. Conversely, assume $[T,M_{ij}]_\xi=0$ for every $M_{ij}\in\mathcal{M}_{ij}$. Write $T=\sum_{i,j=1}^2 T_{ij}$. It follows that $T_{ii}M_{ij}+T_{ji}M_{ij}=\xi(M_{ij}T_{jj}+M_{ij}T_{ji})$. Thus

$$T_{ii}M_{ij}=\xi M_{ij}T_{jj} \tag{1}$$

and $T_{ji}M_{ij}=0$. Noting that $\overline{P_j}=I$, we obtain

$$T_{ji}=0.$$

For every $M_{ii}\in\mathcal{M}_{ii}$, $M_{jj}\in\mathcal{M}_{jj}$, $M_{ii}M_{ij},M_{ij}M_{jj}\in\mathcal{M}_{ij}$ and so $TM_{ii}M_{ij}=\xi M_{ii}M_{ij}T$ and $TM_{ij}M_{jj}=\xi M_{ij}M_{jj}T$. From $[T,M_{ij}]_\xi=0$, it follows that $TM_{ii}M_{ij}=M_{ii}TM_{ij}$, that is $(TM_{ii}-M_{ii}T)M_{ij}=0$. Using $\overline{P_j}=I$ again, we have $T_{ii}M_{ii}-M_{ii}T_{ii}=0$, i.e., $T_{ii}\in\mathcal{Z}_{P_i\mathcal{M}P_i}$. Thus

$$T_{ii}=Z_iP_i$$

for some central element $Z_i\in\mathcal{Z}_{\mathcal{M}}$. Similarly, combining $TM_{ij}M_{jj}=\xi M_{ij}M_{jj}T$ and $[T,M_{ij}]_\xi=0$, we can obtain

$$T_{jj}=Z_jP_j$$

for some central element $Z_j\in\mathcal{Z}_{\mathcal{M}}$. Now equation (1) implies that $(Z_i-\xi Z_j)M_{ij}=0$. From $\overline{P_j}=I$ and M_{ij} is arbitrary, it follows that $(Z_i-\xi Z_j)P_i=0$. Since $Z_i-\xi Z_j\in\mathcal{Z}_{\mathcal{M}}$,

$M(Z_i - \xi Z_j)P_i = (Z_i - \xi Z_j)MP_i = 0$ for all $M \in \mathcal{M}$. By $\overline{P_i} = I$, it follows that $Z_i = \xi Z_j$. So $T = T_{ij} + (\xi P_i + P_j)Z_j \in \mathcal{M}_{ij} + (\xi P_i + P_j)\mathcal{Z}_{\mathcal{M}}$.

3. Proofs of main results

In the following, we are firstly aimed to prove Theorem 1.1.

Proof of Theorem 1.1. In what follows, $\Phi : \mathcal{M} \to \mathcal{M}$ is a nonlinear derivation. We will prove that Φ is additive, that is, for all $T, S \in \mathcal{M}$, $\Phi(T+S) = \Phi(T) + \Phi(S)$. It is clear that $\Phi(0) = \Phi(0)0 + 0\Phi(0) = 0$. Note that $\Phi(P_1P_2) = \Phi(P_1)P_2 + P_1\Phi(P_2) = 0$, multiplying by P_2 from the both sides of this equation, we get $P_2\Phi(P_1)P_2 = 0$. Similarly, multiplying by P_1 from the both sides of this equation, we have $P_1\Phi(P_2)P_1 = 0$. For every $M_{12} \in \mathcal{M}_{12}$, $\Phi(M_{12}) = \Phi(P_1M_{12}) = \Phi(P_1)M_{12} + P_1\Phi(M_{12})$ and so $P_1\Phi(P_1)M_{12} = 0$. Hence $P_1\Phi(P_1)P_1 = 0$. Similarly, from $\Phi(M_{12}) = \Phi(M_{12}P_2)$, one can obtain $P_2\Phi(P_2)P_2 = 0$.

Denote $T_0 = P_1\Phi(P_1)P_2 - P_2\Phi(P_1)P_1$. Define $\Psi : \mathcal{M} \to \mathcal{M}$ by $\Psi(T) = \Phi(T) - [T, T_0]$ for every $T \in \mathcal{M}$. Then it is easy to see that Ψ is also a nonlinear derivation and $\Psi(P_1) = \Psi(P_2) = 0$. Note that for every $T \in \mathcal{M} : T \mapsto [T, T_0]$ is an additive derivation of $\mathcal{M}$. Therefore, without loss of generality, we may assume $\Phi(P_1) = \Phi(P_2) = 0$. Then for every $T_{ij} \in \mathcal{M}_{ij}$, $\Phi(T_{ij}) = P_i\Phi(M_{ij})P_j \in \mathcal{M}_{ij}$ $(i, j = 1, 2)$.

Let T be in $\mathcal{M}$, write $T = T_{11} + T_{12} + T_{21} + T_{22}$. In order to prove the additivity of Φ, we only need to show Φ is additive on $\mathcal{M}_{ij}(1 \le i, j \le 2)$ and $\Phi(T_{11} + T_{12} + T_{21} + T_{22}) = \Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + \Phi(T_{22})$. We will complete the proof by checking two claims.

Claim 1. Φ is additive on $\mathcal{M}_{ij}(1 \le i, j \le 2)$.

Set $T_{ij}, S_{ij}, M_{ij} \in \mathcal{M}_{ij}$. From $(T_{11} + T_{12})M_{12} = T_{11}M_{12}$, it follows that

$$\Phi(T_{11} + T_{12})M_{12} + (T_{11} + T_{12})\Phi(M_{12}) = \Phi(T_{11})M_{12} + T_{11}\Phi(M_{12}).$$

Note that $\Phi(M_{12}) \in \mathcal{M}_{12}$, so $(\Phi(T_{11}+T_{12}) - \Phi(T_{11}))M_{12} = 0$. Then $(\Phi(T_{11}+T_{12}) - \Phi(T_{11}))P_1 = 0$. This implies

$$(\Phi(T_{11} + T_{12}) - \Phi(T_{11}) - \Phi(T_{12}))P_1 = 0.$$

Similarly, from $(T_{11} + T_{12})M_{21} = T_{12}M_{21}$, we have $(\Phi(T_{11} + T_{12}) - \Phi(T_{11}) - \Phi(T_{12}))M_{21} = 0$. Then

$$(\Phi(T_{11} + T_{12}) - \Phi(T_{11}) - \Phi(T_{12}))P_2 = 0.$$

Thus

$$\Phi(T_{11} + T_{12}) = \Phi(T_{11}) + \Phi(T_{12}).$$

Similarly, $\Phi(T_{12} + T_{22}) = \Phi(T_{12}) + \Phi(T_{22})$. Since $T_{12} + S_{12} = (P_1 + T_{12})(P_2 + S_{12})$, we have that

$$\begin{aligned}\Phi(T_{12} + S_{12}) &= \Phi(P_1 + T_{12})(P_2 + S_{12}) + (P_1 + T_{12})\Phi(P_2 + S_{12}) \\ &= \Phi(T_{12}) + \Phi(S_{12}).\end{aligned}$$

In the same way, one can show that $\Phi(T_{21} + S_{21}) = \Phi(T_{21}) + \Phi(S_{21})$. That is, Φ is additive on $\mathcal{M}_{12}, \mathcal{M}_{21}$.

From $(T_{11}+S_{11})M_{12}=T_{11}M_{12}+S_{11}M_{12}$, it follows that

$$\begin{aligned} & \Phi(T_{11}+S_{11})M_{12}+(T_{11}+S_{11})\Phi(M_{12}) \\ = & \Phi(T_{11}M_{12})+\Phi(S_{11}M_{12}) \\ = & \Phi(T_{11})M_{12}+T_{11}\Phi(M_{12})+\Phi(S_{11})M_{12}+S_{11}\Phi(M_{12}). \end{aligned}$$

Thus $(\Phi(T_{11}+S_{11})-\Phi(T_{11})-\Phi(S_{11}))M_{12}=0$. This yields

$$(\Phi(T_{11}+S_{11})-\Phi(T_{11})-\Phi(S_{11}))P_1=0.$$

Note that $\Phi(T_{11}+S_{11})-\Phi(T_{11})-\Phi(S_{11})\in\mathcal{M}_{11}$. So

$$\Phi(T_{11}+S_{11})=\Phi(S_{11})+\Phi(T_{11}).$$

Similarly ,$\Phi(T_{22}+S_{22})=\Phi(T_{22})+\Phi(S_{22})$. That is, Φ is additive on $\mathcal{M}_{11},\mathcal{M}_{22}$, as desired.

Claim 2. $\Phi(T_{11}+T_{12}+T_{21}+T_{22})=\Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22})$

From $(T_{11}+T_{12}+T_{21}+T_{22})M_{12}=(T_{11}+T_{21})M_{12}$, we have

$$\begin{aligned} & \Phi(T_{11}+T_{12}+T_{21}+T_{22})M_{12}+(T_{11}+T_{12}+T_{21}+T_{22})\Phi(M_{12}) \\ = & \Phi(T_{11}M_{12})+\Phi(T_{21}M_{12}) \\ = & \Phi(T_{11})M_{12}+T_{11}\Phi(M_{12})+\Phi(T_{21})M_{12}+T_{21}\Phi(M_{12}). \end{aligned}$$

Then

$$\begin{aligned} & (\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}))M_{12} \\ = & (\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{21}))M_{12}=0. \end{aligned}$$

This gives

$$(\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}))P_1=0.$$

From $(T_{11}+T_{12}+T_{21}+T_{22})M_{21}=(T_{12}+T_{22})M_{21}$, it follows that

$$(\Phi(T_{11}+T_{22}+T_{12}+T_{21})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}))P_2=0.$$

So $\Phi(T_{11}+T_{12}+T_{21}+T_{22})=\Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22})$.

Now, we turn to prove Theorem 1.2.

Proof of Theorem 1.2. We will finish the proof of the Theorem 1.2 by checking several claims.

Claim 1. $\Phi(0)=0$ and there is $T_0\in\mathcal{M}$ such that $\Phi(P_i)=[P_i,T_0]$ $(i=1,2)$.

It is clear that $\Phi(0)=\Phi([0,0]_\xi)=[\Phi(0),0]_\xi+[0,\Phi(0)]_\xi=0$.

For every M_{12},

$$\begin{aligned} \Phi(M_{12}) & =\Phi([P_1,M_{12}]_\xi)=[\Phi(P_1),M_{12}]_\xi+[P_1,\Phi(M_{12})]_\xi \\ & =\Phi(P_1)M_{12}-\xi M_{12}\Phi(P_1)+P_1\Phi(M_{12})-\xi\Phi(M_{12})P_1. \end{aligned} \tag{2}$$

Multiplying by P_1,P_2 from the left and the right in equation (2) respectively, we have

$$P_1\Phi(P_1)P_1M_{12}=\xi M_{12}P_2\Phi(P_1)P_2.$$

That is $[P_1\Phi(P_1)P_1 + P_2\Phi(P_1)P_2, M_{12}]_\xi = 0$. Now Lemma 2.2 yields that $P_1\Phi(P_1)P_1 + P_2\Phi(P_1)P_2 \in (\xi P_1 + P_2)\mathcal{Z}_\mathcal{M}$. For every M_{21},

$$\begin{aligned}\Phi(M_{21}) &= \Phi([P_2, M_{21}]_\xi) = [\Phi(P_2), M_{21}]_\xi + [P_2, \Phi(M_{21})]_\xi \\ &= \Phi(P_2)M_{21} - \xi M_{21}\Phi(P_2) + P_2\Phi(M_{21}) - \xi\Phi(M_{21})P_2.\end{aligned} \tag{3}$$

Multiplying by P_2, P_1 from the left and the right in equation (3) respectively, we obtain

$$P_2\Phi(P_2)P_2M_{21} = \xi M_{21}P_1\Phi(P_2)P_1.$$

That is $[P_2\Phi(P_2)P_2 + P_1\Phi(P_2)P_1, M_{21}]_\xi = 0$. Using Lemma 2.2 again, we get $P_2\Phi(P_2)P_2 + P_1\Phi(P_2)P_1 \in (P_1+\xi P_2)\mathcal{Z}_\mathcal{M}$. Assume $P_1\Phi(P_1)P_1+P_2\Phi(P_1)P_2 = (\xi P_1+P_2)Z_1$ and $P_2\Phi(P_2)P_2+P_1\Phi(P_2)P_1 = (P_1 + \xi P_2)Z_2$, $Z_1, Z_2 \in \mathcal{Z}_\mathcal{M}$. From $[P_1, P_2]_\xi = 0$, it follows that

$$\begin{aligned}&\Phi([P_1, P_2]_\xi) = [\Phi(P_1), P_2]_\xi + [P_1, \Phi(P_2)]_\xi \\ =\ & \Phi(P_1)P_2 - \xi P_2\Phi(P_1) + P_1\Phi(P_2) - \xi\Phi(P_2)P_1 \\ =\ & (1-\xi)P_1\Phi(P_2)P_1 + (1-\xi)P_2\Phi(P_1)P_2 + P_1\Phi(P_1)P_2 \\ & +P_1\Phi(P_2)P_2 - \xi P_2\Phi(P_2)P_1 - \xi P_2\Phi(P_1)P_1 \\ =\ & 0.\end{aligned}$$

Then

$$P_1\Phi(P_2)P_1 = P_2\Phi(P_1)P_2 = P_1\Phi(P_1)P_2 + P_1\Phi(P_2)P_2 = P_2\Phi(P_1)P_1 + P_2\Phi(P_2)P_1 = 0. \tag{4}$$

A direct computation shows that $[(\xi P_1 + P_2)Z_1, P_2]_\xi = [P_1\Phi(P_1)P_1 + P_2\Phi(P_1)P_2, P_2]_\xi = 0$. And so $(1-\xi)P_2Z_1 = 0$. Then $Z_1MP_2 = 0$ for all $M \in \mathcal{M}$. Noting that $\overline{P}_2 = I$, we have $Z_1 = 0$. That is $P_1\Phi(P_1)P_1 + P_2\Phi(P_1)P_2 = 0$. Similarly, $P_2\Phi(P_2)P_2 + P_1\Phi(P_2)P_1 = 0$. By (4), $\Phi(P_1) + \Phi(P_2) = 0$. Denote $T_0 = P_1\Phi(P_1)P_2 - P_2\Phi(P_1)P_1$. Then it is easy to check that T_0 is the desired.

Obviously, $T \mapsto [T, T_0]$ is an additive derivation. Without loss of generality, we may assume that $\Phi(P_1) = \Phi(P_2) = 0$.

If Φ is additive, then $\Phi(I) = \Phi(P_1) + \Phi(P_2) = 0$. $\Phi((1-\xi)T) = \Phi([I, T]_\xi) = [I, \Phi(T)]_\xi = (1-\xi)\Phi(T)$ for all $T \in \mathcal{M}$. So $\Phi(\xi T) = \xi\Phi(T)$ for all $T \in \mathcal{M}$. Taking $T, S \in \mathcal{M}$ and noting that $(1-\xi)[S, T]_{-1} = [S, T]_\xi + [T, S]_\xi$, we obtain that

$$\begin{aligned}&\Phi((1-\xi)[S, T]_{-1}) = \Phi([S, T]_\xi) + \Phi([T, S]_\xi) \\ =\ & \Phi(S)T - \xi T\Phi(S) + S\Phi(T) - \xi\Phi(T)S + \Phi(T)S - \xi S\Phi(T) + T\Phi(S) - \xi\Phi(S)T \\ =\ & (1-\xi)(\Phi(S)T + S\Phi(T) + \Phi(T)S + T\Phi(S)).\end{aligned}$$

Note that $\Phi((1-\xi)T) = (1-\xi)\Phi(T)$ for all $T \in \mathcal{M}$, it follows that

$$\Phi([S, T]_{-1}) = [\Phi(S), T]_{-1} + [S, \Phi(T)]_{-1}$$

for all $T, S \in \mathcal{M}$. Hence Φ is an additive Jordan derivation. By [5], Φ is an additive derivation which is the conclusion of our Theorem 1.2. Now we only need to show Φ is additive. For every $T \in \mathcal{M}$, it has the form $T = T_{11} + T_{12} + T_{21} + T_{22}$. Just like the proof of Theorem 1.1,

we will show Φ is additive on $\mathcal{M}_{ij}(1 \leq i,j \leq 2)$ and $\Phi(T_{11}+T_{12}+T_{21}+T_{22}) = \Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + \Phi(T_{22})$. We divide the proof into several steps.

Claim 2. $\Phi(M_{ij}) \in \mathcal{M}_{ij}$ for every $M_{ij} \in \mathcal{M}_{ij}$ $(1 \leq i \neq j \leq 2)$.

We only treat the case $i = 1, j = 2$. The other case can be treated similarly. Noting $[P_1, M_{12}]_\xi = M_{12}$, we have

$$\begin{aligned}\Phi(M_{12}) = \Phi([P_1, M_{12}]_\xi) &= [\Phi(P_1), M_{12}]_\xi + [P_1, \Phi(M_{12})]_\xi \\ &= [P_1, \Phi(M_{12})]_\xi = P_1\Phi(M_{12}) - \xi\Phi(M_{12})P_1.\end{aligned}$$

Then

$$P_2\Phi(M_{12})P_2 = P_1\Phi(M_{12})P_1 = 0. \tag{5}$$

Furthermore, $P_2\Phi(M_{12})P_1 = 0$, if $\xi \neq -1$, i.e., $\Phi(M_{12}) \in \mathcal{M}_{12}$.

Next we treat the case $\xi = -1$. For every M_{11},

$$\begin{aligned}\Phi(M_{11}M_{12}) = \Phi([M_{11}, M_{12}]_{-1}) &= [\Phi(M_{11}), M_{12}]_{-1} + [M_{11}, \Phi(M_{12})]_{-1} \\ &= \Phi(M_{11})M_{12} + M_{12}\Phi(M_{11}) + M_{11}\Phi(M_{12}) + \Phi(M_{12})M_{11}.\end{aligned}$$

By (5), we have

$$P_2\Phi(M_{11}M_{12})P_1 = \Phi(M_{12})M_{11}.$$

Then for every N_{11}, $P_2\Phi(N_{11}M_{11}M_{12})P_1 = \Phi(M_{12})N_{11}M_{11}$. On the other hand,

$$P_2\Phi(N_{11}M_{11}M_{12})P_1 = \Phi(M_{11}M_{12})N_{11} = \Phi(M_{12})M_{11}N_{11}.$$

Thus $\Phi(M_{12})[N_{11}, M_{11}] = 0$. For every R_{11},

$$\Phi(M_{12})R_{11}[N_{11}, M_{11}] = P_2\Phi(R_{11}M_{12})P_1[N_{11}, M_{11}] = 0.$$

By Lemma 2.1(ii), $\Phi(M_{12})P_1 = 0$ which finishes the proof.

Claim 3. $\Phi(M_{ii}) \in \mathcal{M}_{ii}$ for every $M_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$.

Proof. Without loss of generality, we only treat the case $i = 1$.

$$\begin{aligned}\Phi(P_1) \quad &= \Phi([I, \tfrac{1}{1-\xi}P_1]_\xi) = [\Phi(I), \tfrac{1}{1-\xi}P_1]_\xi + [I, \Phi(\tfrac{1}{1-\xi}P_1)]_\xi \\ &= \tfrac{1}{1-\xi}\Phi([I, P_1]_\xi) + [I, \Phi(\tfrac{1}{1-\xi}P_1)]_\xi \\ &= \tfrac{1}{1-\xi}\Phi((1-\xi)P_1) + (1-\xi)\Phi(\tfrac{1}{1-\xi}P_1) = 0.\end{aligned}$$

Note that $\Phi((1-\xi)P_1) = \Phi([P_1, P_1]_\xi) = 0$, so $\Phi(\frac{1}{1-\xi}P_1) = 0$.

$$\begin{aligned}\Phi(M_{11}) = \Phi([\frac{1}{1-\xi}P_1, M_{11}]_\xi) &= [\frac{1}{1-\xi}P_1, \Phi(M_{11})]_\xi \\ &= \frac{1}{1-\xi}(P_1\Phi(M_{11}) - \xi\Phi(M_{11})P_1).\end{aligned}$$

This implies $\Phi(M_{11}) \in \mathcal{M}_{11}$.

Claim 4. For every T_{ii}, T_{ji} and T_{ij} $(1 \leq i \neq j \leq 2)$, $\Phi(T_{ii} + T_{ij}) = \Phi(T_{ii}) + \Phi(T_{ij})$, $\Phi(T_{ii} + T_{ji}) = \Phi(T_{ii}) + \Phi(T_{ji})$.

Assume $i = 1, j = 2$. For every $M_{12} \in \mathcal{M}_{12}$, $[T_{11} + T_{12}, M_{12}]_\xi = [T_{11}, M_{12}]_\xi$, by Claim 2,

$$[\Phi(T_{11} + T_{12}), M_{12}]_\xi + [T_{11} + T_{12}, \Phi(M_{12})]_\xi = [\Phi(T_{11}), M_{12}]_\xi + [T_{11}, \Phi(M_{12})]_\xi,$$

$$[\Phi(T_{11}+T_{12})-\Phi(T_{11}), M_{12}]_\xi = 0.$$

From Lemma 2.2,

$$\Phi(T_{11}+T_{12})-\Phi(T_{11}) = P_1(\Phi(T_{11}+T_{12})-\Phi(T_{11}))P_2 + (\xi P_1 + P_2)Z$$

for some central element $Z \in Z_{\mathcal{M}}$. By computing,

$$\begin{aligned}\Phi(T_{12}) &= \Phi([P_1,[T_{11}+T_{12},P_2]_\xi]_\xi)\\ &= [P_1,[\Phi(T_{11}+T_{12}),P_2]_\xi]_\xi\\ &= P_1\Phi(T_{11}+T_{22})P_2 + \xi^2 P_2\Phi(T_{11}+T_{22}).\end{aligned}$$

From Claim 2 and Claim 3, we know that $\Phi(T_{12}) = P_1\Phi(T_{11}+T_{12})P_2$ and $P_1\Phi(T_{11})P_2 = 0$. Thus

$$\Phi(T_{11}+T_{12})-\Phi(T_{11}) = \Phi(T_{12}) + (\xi P_1 + P_2)Z.$$

Note that

$$\begin{aligned}\Phi([T_{11}+T_{12},P_2]_\xi) &= [\Phi(T_{11}+T_{12}),P_2]_\xi\\ &= [\Phi(T_{11})+\Phi(T_{12})+(\xi P_1+P_2)Z,P_2]_\xi.\end{aligned}$$

On the other hand, $\Phi([T_{11}+T_{12},P_2]_\xi) = \Phi([T_{12},P_2]_\xi) = [\Phi(T_{12}),P_2]_\xi$. Combining this with Claim 3, we have $[(\xi P_1+P_2)Z,P_2]_\xi = 0$ and so $ZP_2 = 0$ which implies $Z = 0$. Similarly, $\Phi(T_{11}+T_{21}) = \Phi(T_{11})+\Phi(T_{21})$. The rest goes similarly.

Claim 5. Φ is additive on $\mathcal{M}_{12}$ and $\mathcal{M}_{21}$.

Let $T_{12}, S_{12} \in \mathcal{M}_{12}$. Since $T_{12}+S_{12} = [P_1+T_{12},P_2+S_{12}]_\xi$, we have that

$$\begin{aligned}&\Phi(T_{12}+S_{12}) = [\Phi(P_1+T_{12}),P_2+S_{12}]_\xi + [P_1+T_{12},\Phi(P_2+S_{12})]_\xi\\ =\;& [\Phi(P_1)+\Phi(T_{12}),P_2+S_{12}]_\xi + [P_1+T_{12},\Phi(P_2)+\Phi(S_{12})]_\xi\\ =\;& \Phi(T_{12})+\Phi(S_{12}).\end{aligned}$$

Similarly, Φ is additive on $\mathcal{M}_{21}$.

Claim 6. For every $T_{11} \in \mathcal{M}_{11}$, $T_{22} \in \mathcal{M}_{22}$, $\Phi(T_{11}+T_{22}) = \Phi(T_{11})+\Phi(T_{22})$.

For every $M_{12} \in \mathcal{M}_{12}$, $[T_{11}+T_{22},M_{12}]_\xi = T_{11}M_{12} - \xi M_{12}T_{22}$. From Claim 5, it follows that

$$\begin{aligned}&[\Phi(T_{11}+T_{22}),M_{12}]_\xi + [T_{11}+T_{22},\Phi(M_{12})]_\xi = \Phi([T_{11}+T_{22},M_{12}]_\xi)\\ =\;& \Phi(T_{11}M_{12}) + \Phi(-\xi M_{12}T_{22}) = \Phi([T_{11},M_{12}]_\xi) + \Phi([T_{22},M_{12}]_\xi)\\ =\;& [\Phi(T_{11}),M_{12}]_\xi + [T_{11},\Phi(M_{12})]_\xi + [\Phi(T_{22}),M_{12}]_\xi + [T_{22},\Phi(M_{12})]_\xi.\end{aligned}$$

Thus $[\Phi(T_{11}+T_{22})-\Phi(T_{11})-\Phi(T_{22}),M_{12}]_\xi = 0$. By Lemma 2.2,

$$\Phi(T_{11}+T_{22})-\Phi(T_{11})-\Phi(T_{22}) \in \mathcal{M}_{12} + (\xi P_1+P_2)\mathcal{Z}_{\mathcal{M}}.$$

On the other hand, $[T_{11}+T_{22},\frac{P_1}{1-\xi}]_\xi = T_{11}$. From the proof of Claim 3, one can see $\Phi(\frac{P_1}{1-\xi}) = 0$. Hence $[\Phi(T_{11}+T_{22}),\frac{P_1}{1-\xi}]_\xi = \Phi(T_{11})$, i.e., $(1-\xi)\Phi(T_{11}) = \Phi(T_{11}+T_{22})P_1 - \xi P_1\Phi(T_{11}+T_{22})$. Multiplying by P_1 and P_2 from the left and the right in the above equation, we have $P_1\Phi(T_{11}+T_{22})P_2 = 0$. So

$$\Phi(T_{11}+T_{22})-\Phi(T_{11})-\Phi(T_{22}) = (\xi P_1+P_2)Z$$

for some central element $Z \in Z_{\mathcal{M}}$. Combining $\Phi(P_1) = 0$ and Claim 3, we conclude

$$\begin{aligned} & \Phi([T_{11}, P_1]_\xi) = \Phi([T_{11} + T_{22}, P_1]_\xi) \\ = \; & [\Phi(T_{11} + T_{22}), P_1]_\xi + [T_{11} + T_{22}, \Phi(P_1)]_\xi \\ = \; & [\Phi(T_{11}) + (\xi P_1 + P_2)Z, P_1]_\xi. \end{aligned}$$

Thus $[(\xi P_1 + P_2)Z, P_1]_\xi = 0$ which implies $Z = 0$. This gives $\Phi(T_{11} + T_{22}) = \Phi(T_{11}) + \Phi(T_{22})$.

Claim 7. For every $T_{ii}, S_{ii} \in \mathcal{M}_{ii}$ $(i = 1, 2)$, $\Phi(T_{ii} + S_{ii}) = \Phi(T_{ii}) + \Phi(S_{ii})$.

Assume $i = 1$. For every $M_{12} \in \mathcal{M}_{12}$, $[T_{11} + S_{11}, M_{12}]_\xi = T_{11}M_{12} + S_{11}M_{12}$. From Claim 5, it follows that

$$\begin{aligned} & [\Phi(T_{11} + S_{11}), M_{12}]_\xi + [T_{11} + S_{11}, \Phi(M_{12})]_\xi = \Phi([T_{11} + S_{11}, M_{12}]_\xi) \\ = \; & \Phi(T_{11}M_{12}) + \Phi(S_{11}M_{12}) = \Phi([T_{11}, M_{12}]_\xi) + \Phi([S_{11}, M_{12}]_\xi) \\ = \; & [\Phi(T_{11}), M_{12}]_\xi + [T_{11}, \Phi(M_{12})]_\xi + [\Phi(S_{11}), M_{12}]_\xi + [S_{11}, \Phi(M_{12})]_\xi. \end{aligned}$$

Thus $[\Phi(T_{11} + S_{11}) - \Phi(T_{11}) - \Phi(S_{11}), M_{12}]_\xi = 0$. By Lemma 2.2,

$$\Phi(T_{11} + S_{11}) - \Phi(T_{11}) - \Phi(S_{11}) \in \mathcal{M}_{12} + (\xi P_1 + P_2)\mathcal{Z}.$$

On the other hand, Claim 3 tells us that $P_1(\Phi(T_{11} + S_{11}) - \Phi(T_{11}) - \Phi(S_{11}))P_2 = 0$. So

$$\Phi(T_{11} + S_{11}) - \Phi(T_{11}) - \Phi(S_{11}) = (\xi P_1 + P_2)Z$$

for some $Z \in \mathcal{Z}_{\mathcal{M}}$. This further indicates

$$\begin{aligned} 0 &= \Phi([T_{11} + S_{11}, P_2]_\xi) = [\Phi(T_{11} + S_{11}), P_2]_\xi \\ &= [\Phi(T_{11}) + \Phi(S_{11}) + (\xi P_1 + P_2)Z, P_2]_\xi \\ &= [(\xi P_1 + P_2)Z, P_2]_\xi. \end{aligned}$$

Then $P_2 Z = 0$, consequently, $Z = 0$. That is, Φ is additive on $\mathcal{M}_{11}$. Similarly, Φ is additive on $\mathcal{M}_{22}$.

Claim 8. For every T_{ii}, T_{jj}, T_{ij}, $(1 \le i \ne j \le 2)$ $\Phi(T_{ii}+T_{jj}+T_{ij}) = \Phi(T_{ii})+\Phi(T_{jj})+\Phi(T_{ij})$.

Assume $i = 1, j = 2$. For every $M_{12} \in \mathcal{M}_{12}$, $[T_{11} + T_{22} + T_{12}, M_{12}]_\xi = [T_{11} + T_{22}, M_{12}]_\xi$. By Claim 6, it follows that

$$[\Phi(T_{11}+T_{22}+T_{12}), M_{12}]_\xi+[T_{11}+T_{22}+T_{12}, \Phi(M_{12})]_\xi = [\Phi(T_{11})+\Phi(T_{22}), M_{12}]_\xi+[T_{11}+T_{22}, \Phi(M_{12})]_\xi.$$

Thus $[\Phi(T_{11} + T_{22} + T_{12}) - \Phi(T_{11}) - \Phi(T_{22}), M_{12}]_\xi = 0$. From Lemma 2.2 and Claim 3, we obtain

$$\begin{aligned} & \Phi(T_{11} + T_{22} + T_{12}) - \Phi(T_{11}) - \Phi(T_{22}) \\ = \; & P_1(\Phi(T_{11} + T_{22} + T_{12}) - \Phi(T_{11}) - \Phi(T_{22}))P_2 + (\xi P_1 + P_2)Z \\ = \; & P_1\Phi(T_{11} + T_{22} + T_{12})P_2 + (\xi P_1 + P_2)Z \end{aligned}$$

for some central element Z. A direct computation shows that

$$\begin{aligned}\Phi(T_{12}) &= \Phi(P_1(T_{11}+T_{22}+T_{12})P_2)\\ &= \Phi([P_1,[T_{11}+T_{22}+T_{12},P_2]_\xi]_\xi)\\ &= [P_1,[\Phi([T_{11}+T_{22}+T_{12},P_2]_\xi)]_\xi\\ &= P_1\Phi(T_{11}+T_{22}+T_{12})P_2.\end{aligned}$$

Thus

$$\Phi(T_{11}+T_{22}+T_{12}) = \Phi(T_{11})+\Phi(T_{22})+\Phi(T_{12})+(\xi P_1+P_2)Z.$$

It is easy to see

$$\begin{array}{l}[\Phi(T_{11}+T_{22}+T_{12}),P_2]_\xi = \Phi([T_{11}+T_{22}+T_{12},P_2]_\xi)\\ = \Phi([T_{12}+T_{22},P_2]_\xi) = [\Phi(T_{12})+\Phi(T_{22}),P_2]_\xi\end{array}.$$

Then $[(\xi P_1+P_2)Z,P_2]=0$, $ZP_2=0$ which implies $Z=0$. That is $\Phi(T_{11}+T_{22}+T_{12}) = \Phi(T_{11})+\Phi(T_{22})+\Phi(T_{12})$. The rest goes similarly.

Claim 9. For every $T_{11},T_{12},T_{21},T_{22}$,

$$\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}) \in (\xi P_1+P_2)\mathcal{Z}_\mathcal{M}\cap(P_1+\xi P_2)\mathcal{Z}_\mathcal{M}.$$

Consequently, if $\xi\neq -1$, $\Phi(T_{11}+T_{12}+T_{21}+T_{22}) = \Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22})$.

For every $M_{12}\in\mathcal{M}_{12}$, $[T_{11}+T_{12}+T_{21}+T_{22},M_{12}]_\xi = [T_{11}+T_{21}+T_{22},M_{12}]_\xi$. From Claim 8, it follows that

$$\begin{array}{rl} & [\Phi(T_{11}+T_{12}+T_{21}+T_{22}),M_{12}]_\xi + [T_{11}+T_{12}+T_{21}+T_{22},\Phi(M_{12})]_\xi\\ = & [\Phi(T_{11})+\Phi(T_{21})+\Phi(T_{22}),M_{12}]_\xi + [T_{11}+T_{21}+T_{22},\Phi(M_{12})]_\xi.\end{array}$$

Thus

$$[\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{21})-\Phi(T_{22}),M_{12}]_\xi = 0.$$

Since $\Phi(T_{12})\in\mathcal{M}_{12}$, we have

$$[\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}),M_{12}]_\xi = 0.$$

Similarly, from $[T_{11}+T_{12}+T_{21}+T_{22},M_{21}]_\xi = [T_{11}+T_{12}+T_{22},M_{21}]_\xi$, we can obtain

$$[\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}),M_{21}]_\xi = 0.$$

From Lemma 2.2, it follows that

$$\Phi(T_{11}+T_{12}+T_{21}+T_{22})-\Phi(T_{11})-\Phi(T_{12})-\Phi(T_{21})-\Phi(T_{22}) \in (\xi P_1+P_2)\mathcal{Z}_\mathcal{M}\cap(P_1+\xi P_2)\mathcal{Z}_\mathcal{M}.$$

Note that if $\xi\neq -1$, $(\xi P_1+P_2)\mathcal{Z}_\mathcal{M}\cap(P_1+\xi P_2)\mathcal{Z}_\mathcal{M} = \{0\}$. Thus

$$\Phi(T_{11}+T_{12}+T_{21}+T_{22}) = \Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22}).$$

Claim 10. If $\xi=-1$, $\Phi(T_{11}+T_{12}+T_{21}+T_{22}) = \Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22})$ holds true, too.

By Claim 9, we may assume $\Phi(T_{12}+T_{21}) = \Phi(T_{12})+\Phi(T_{21})+(-P_1+P_2)Z_1$, $\Phi(T_{11}+T_{12}+T_{21}) = \Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+(-P_1+P_2)Z_2$ and $\Phi(T_{11}+T_{12}+T_{21}+T_{22}) = \Phi(T_{11})+\Phi(T_{12})+\Phi(T_{21})+\Phi(T_{22})+(-P_1+P_2)Z_3$. The following is devoted to showing

$Z_1 = Z_2 = Z_3 = 0$. Since $\Phi(T_{12} + T_{21}) = \Phi([T_{12} + T_{21}, P_1]_{-1}) = [\Phi(T_{12} + T_{21}), P_1]_{-1}$, substituting $\Phi(T_{12} + T_{21}) = \Phi(T_{12}) + \Phi(T_{21}) + (-P_1 + P_2)Z_1$ into above equation, we have $(-P_1 + P_2)Z_1 = [(-P_1 + P_2)Z_1, P_1]_{-1} = -2P_1Z_1$. Then $Z_1P_1 = Z_1P_2 = 0$ and so $Z_1 = 0$. From

$$\begin{aligned} & [\Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + (-P_1 + P_2)Z_2, P_2]_{-1} \\ = \ & [\Phi(T_{11} + T_{12} + T_{21}), P_2]_{-1} = \Phi([T_{11} + T_{12} + T_{21}, P_2]_{-1}) \\ = \ & \Phi(T_{12} + T_{21}) = \Phi(T_{12}) + \Phi(T_{21}) \\ = \ & [\Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}), P_2]_{-1}, \end{aligned}$$

it follows that $[(-P_1 + P_2)Z_2, P_2] = 0$. Thus $Z_2 = 0$. At last,

$$\begin{aligned} & [\Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + \Phi(T_{22}) + (-P_1 + P_2)Z_3, P_1]_{-1} \\ = \ & [\Phi(T_{11} + T_{12} + T_{21} + T_{22}), P_1]_{-1} = \Phi([T_{11} + T_{12} + T_{21} + T_{22}, P_1]_{-1}) \\ = \ & \Phi([T_{11} + T_{12} + T_{21}, P_1]_{-1}) = [\Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + \Phi(T_{22}), P_1]_{-1}. \end{aligned}$$

So $[(-P_1 + P_2)Z_3, P_1]_{-1} = 0$ which implies $Z_3 = 0$. Hence $\Phi(T_{11} + T_{12} + T_{21} + T_{22}) = \Phi(T_{11}) + \Phi(T_{12}) + \Phi(T_{21}) + \Phi(T_{22})$, as desired.

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